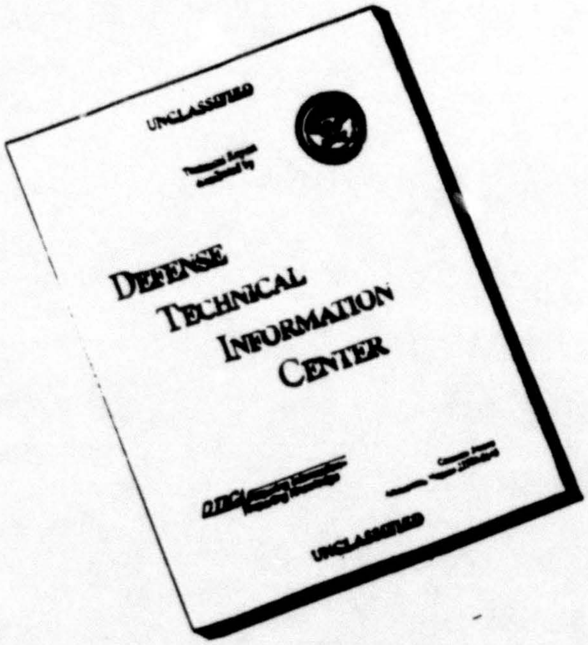
NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

03105, 2794-13-T

THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS

DEPARTMENT OF PHILOSOPHY
LOGIC OF COMPUTERS GROUP

Technical Note

On a Problem of Tarski

J. RICHARD BÜCHI

Under Contract With:

Department of the Army
U. S. Army Signal Supply Agency
Contract No. DA 36-039-sc-78057
Fort Monmouth, New Jersey

and

Department of the Navy
Office of Naval Research
Contract No. Nonr-1224 (21)
Washington, D.C.

Administered by: *July 1960*

THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE • ANN ARBOR

THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Philosophy
Logic of Computers Group

Technical Note

ON A PROBLEM OF TARSKI

J. Richard Büchi

UMRI Projects 2794 and 03105

under contract with:

DEPARTMENT OF THE ARMY
U. S. ARMY SIGNAL SUPPLY AGENCY
CONTRACT NO. DA 36-039-sc-78057
FORT MONMOUTH, NEW JERSEY
and
DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. Nonr-1224(21)
WASHINGTON, D. C.

administered by:

THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE ANN ARBOR

July 1960

On a problem of Tarski[1]

by J. Richard Büchi

Let SC (sequential calculus) be the interpreted formalism which makes use of individual variables t, x, y, z, ... ranging over natural numbers, monadic predicate variables q( ), r( ), s( ), i( ), j( ), ... ranging over arbitrary sets of natural numbers, the individual symbol o for zero, the function symbol ʼ denoting the successor function, propositional connectives, and quantifiers for both types of variables.

The purpose of this ~~note is to outline~~ an effective method for deciding truth of sentences in ~~SC~~*. This, according to R. M. Robinson** ~~[10],~~ answers a question of Tarski's. In addition, a rather complete understanding of definability in sequential calculus ~~SC will be~~ obtained.

* in sequential calculus is outlined.

** [Proc. Am. Math. Soc. 9:238-242, 1958],

1. Notations. $\underset{\sim}{i}$ denotes a n-tuple of predicate variables. Expressions like $A[\underset{\sim}{i}(o)]$, $B[\underset{\sim}{i}(t), \underset{\sim}{i}(t')]$ denote propositional formulas in the indicated constituents. Σ_n, π_n, denote the classes of formulas of SC of the following type,

$$\Sigma_1 \quad : \quad (\exists \underset{\sim}{r}) \cdot A[\underset{\sim}{r}(o)] \wedge (\forall t)\, B[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \wedge (\exists t)\, C[\underset{\sim}{r}(t)]$$

$$\pi_1 \quad : \quad (\forall \underset{\sim}{r}) \cdot A[\underset{\sim}{r}(o)] \vee (\exists t)\, B[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \vee (\forall t)\, C[\underset{\sim}{r}(t)]$$

$$\Sigma_{n+1} \quad : \quad (\exists \underset{\sim}{r}) \cdot \underline{F}(\underset{\sim}{i}, \underset{\sim}{r}) \qquad , \text{ whereby } \underline{F} \in \pi_n$$

$$\pi_{n+1} \quad : \quad (\forall \underset{\sim}{r}) \cdot \underline{F}(\underset{\sim}{i}, \underset{\sim}{r}) \qquad , \text{ whereby } \underline{F} \in \Sigma_n$$

The quantifiers $(\exists t)_x^y\, \underline{A}(t)$ for $(\exists t)\ [x \leq t < y \wedge \underline{A}(t)]$, $(\forall t)_x^y\, A(t)$ for $(\forall t)[x \leq t < y \supset \underline{A}(t)]$, $(\exists^\omega t)\ \underline{A}(t)$ for $(\forall x)(\exists t)\ [x < t \wedge \underline{A}(t)]$, $(\forall_\omega t)\ \underline{A}(t)$ for $(\exists x)(\forall t)[x < t]\ \ \underline{A}(t)$, $(\exists j)_\omega\, A(j)$ for $(\exists j)\ [(\exists^\omega t) j(t) \wedge \underline{A}(j)]$ can be defined in SC. The classes Σ_1^ω and π_1^ω of formulas are defined as follows,

$$\Sigma_1^\omega \quad : \quad (\exists r) \cdot A[\underset{\sim}{r}(o)] \wedge (\forall t) B[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \wedge (\exists^\omega t)\ c[\underset{\sim}{r}(t)]$$

$$\pi_1^\omega \quad : \quad (\forall r) \cdot A[\underset{\sim}{r}(o)] \vee (\exists t)\ B[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \vee (\forall_\omega t)\ C[\underset{\sim}{r}(t)]$$

---

1. The author is much indebted to Dr. J. B. Wright. The work was done under a grant from the National Science Foundation to the Logic of Computers Group, and with additional assistance through contracts with the Office of Naval Research, Office of Ordnance Research, and the Army Signal Corps.

Let $\underset{\sim}{i}$ be a k-tuple of predicates. The 2^k states of $\underset{\sim}{i}$ are the k-tuples of truth-values. $\underset{\sim}{i}$ may be viewed as an infinite sequence $\underset{\sim}{i}(o)\ \underset{\sim}{i}(1)\ \underset{\sim}{i}(2) \ldots$ of states. The variables $\underline{u}$, $\underline{v}$, $\underline{w}$, ... will be used for words (i.e., finite sequences) of states. $\underline{u}\frown\underline{v}$ denotes the result of juxtaposing $\underline{u}$ and $\underline{v}$. A congruence-relation is an equivalence relation $\underline{u} \backsim \underline{v}$ on words such that $\underline{u} \backsim \underline{v}$ implies $\underline{u}\frown\underline{w} \backsim \underline{v}\frown\underline{w}$ and $\underline{w}\frown\underline{u} \backsim \underline{w}\frown\underline{v}$. An equivalence relation is of finite rank if it partitions the words into finitely many classes.

Also the following classes of formulas will play an essential role,

$$\Sigma_{reg} \ : \ (\exists \underset{\sim}{r}) \cdot A[\underset{\sim}{r}(x)] \wedge (\forall t_x^y\, B[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \wedge C[\underset{\sim}{r}(y)]$$

$$\pi_{reg} \ : \ (\forall r) \cdot A[\underset{\sim}{r}(x)] \vee (\exists t_x^y\, B[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \vee C[\underset{\sim}{r}(y)].$$

These may be called regular formulas. Note that for a regular formula, $\underline{R}(\underset{\sim}{i},x,y)$ depends only on the word $\underset{\sim}{i}(x)\ \underset{\sim}{i}(x+1) \ldots \underset{\sim}{i}(y-1)$. If $\underline{\underline{R}}$ is the set of all words $\underset{\sim}{i}(o)\ \underset{\sim}{i}(1) \ldots \underset{\sim}{i}(h)$ such that $\underline{R}(\underset{\sim}{i},o,h+1)$, then the formula $\underline{R}(\underset{\sim}{i},x,y)$ is said to determine the set of words $\underline{\underline{R}}$.

2. A fundamental lemma on infinite sequences. The working of the decision-method for SC is based on induction and a rather more sophisticated property of infinity, namely Theorem A of Ramsey [9].[2] Essential parts of this theorem can actually be formulated in SC, in the form of a surprising assertion about the division of infinite sequences into consecutive finite parts.

Lemma 1. Let $\underset{\sim}{i}$ be any k-tuple of predicates, and let $\underline{\underline{E}}_o, \ldots, \underline{\underline{E}}_n$ be a partition of all words on states of $\underset{\sim}{i}$ into finitely many classes. Then there exists a division $\underset{\sim}{i}(o)\ \underset{\sim}{i}(1) \ldots \underset{\sim}{i}(x_1-1)$, $\underset{\sim}{i}(x_1)\ \underset{\sim}{i}(x_1+1) \ldots \underset{\sim}{i}(x_2-1)$, $\underset{\sim}{i}(x_2)\ \underset{\sim}{i}(x_2+1) \ldots \underset{\sim}{i}(x_3-1)$, ... of $\underset{\sim}{i}$ such that all words $\underset{\sim}{i}(x_p)\ \underset{\sim}{i}(x_p+1) \ldots \underset{\sim}{i}(x_q-1)$ belong to one and the same of the classes $\underline{\underline{E}}_o, \ldots, \underline{\underline{E}}_n$.

Proof: Assume i, $\underline{\underline{E}}_o, \ldots, \underline{\underline{E}}_n$ are as supposed in lemma 1. For $o \leq c \leq n$ let P_c consist of all $\{y_1,y_2\}$ such that $y_1 < y_2$ and $\underset{\sim}{i}(y_1)\ \underset{\sim}{i}(y_1+1) \ldots \underset{\sim}{i}(y_2=1) \varepsilon \underline{\underline{E}}_c$. Then $P_o, \ldots, P_n$ clearly is a partition of all 2-element sets of natural numbers. By Ramsey's theorem A it follows that there is an infinite sequence $x_1 < x_2 < x_3 < \ldots$ and a $o \leq c \leq n$, such that $\{x_p,x_q\} \varepsilon P_c$ for all $x_p < x_q$. By definition of P_c this yields the conclusion of lemma 1.

3. Automata-theory. The following concepts and results are borrowed from the theory of finite automata, and play a very essential role in the study of SC.

2. The usefulness of the "Unendlichkeitslemma" of König [5] (also known as "fan-theorem" in its intuitionistic version) in related problems of automata-theory was first observed by Dr. J. B. Wright. Because of its affinity to König's lemma the present application of Ramsey's theorem was suggested.

The reader is referred to Büchi [1], where some of the details are carried out in similar form, and where further references to the mathematical literature on automata are given

Lemma 2. The following are equivalent conditions on a set $\underline{\underline{R}}$ of words:

1) $\underline{\underline{R}}$ is regular.

2) $\underline{\underline{R}} = \underline{\underline{E}}_1 \cup \cdots \cup \underline{\underline{E}}_n$, whereby $\underline{\underline{E}}_1, \ldots, \underline{\underline{E}}_n$ are some of the congruence classes modulo a congruence relation of finite rank on words.

3) $\underline{\underline{R}}$ is the set of words determined by a formula $\underline{F}(\underset{\sim}{i},x,y)$ belonging to Σ_{reg}.

4) $\underline{\underline{R}}$ is the set of words determined by a formula $\underline{G}(\underset{\sim}{i},x,y)$ belonging to π_{reg}.

5) There is an "automata-recursion" $\underset{\sim}{r}(o) \equiv I$, $\underset{\sim}{r}(t') \equiv J[\underset{\sim}{i}(t), \underset{\sim}{r}(t)]$ and an "output" $U[\underset{\sim}{r}(t)]$ such that a word $\underset{\sim}{i}(o)\ \underset{\sim}{i}(1) \ldots \underset{\sim}{i}(x-1)$ belongs to $\underline{\underline{R}}$ just in case the recursion produces an $\underset{\sim}{r}(x)$ such that $U[r(x)]$ holds.

The reader who is not familiar with the concept of regularity of Kleene [4] may take $1 \leftrightarrow 2$ of the lemma as its definition (note the analogy to ultimately periodic sets of natural numbers). $3 \leftrightarrow 5$ is shown in essence by Myhill's "subset-construction" [6]; nearly in the present form the details are carried out in Büchi [1], lemma 7. By dualization one gets $4 \rightarrow 5$. The assertions $5 \rightarrow 3$. and $5 \rightarrow 4$ are trivial. A proof of $2 \leftrightarrow 3$ is contained essentially in Rabin and Scott [8].

Lemma 3. If the formulas $\underline{R}(\underset{\sim}{i},x,y)$, $\underline{S}(\underset{\sim}{i},x,y)$ determine regular sets of words, then so do the formulas $(\exists t)_x^y\ \underline{R}(\underset{\sim}{i},t,y)$, $(\forall t)_x^y\ \underline{R}(\underset{\sim}{i},t,y)$, $\underline{R}(\underset{\sim}{i},x,y)\quad \underline{S}(\underset{\sim}{i},x,y)$, $\underline{R}(\underset{\sim}{i},x,y) \vee \underline{S}(\underset{\sim}{i},x,y)$, and $\sim\underline{R}(\underset{\sim}{i},x,y)$.

This follows by lemma 2 and Skolem's method of replacing bounded quantifiers by recursions.

Lemma 4. If the formula $\underline{R}(\underset{\sim}{i},x,y)$ determines a regular set, then one can find formulas $\underline{D}(\underset{\sim}{i})$ and $\underline{E}(\underset{\sim}{i})$ in Σ_1 (in π_1) such that $\underline{D}(\underset{\sim}{i}) \equiv (\forall t)\ \underline{R}(\underset{\sim}{i},o,t)$ and $\underline{E}(\underset{\sim}{i}) \equiv (\exists t)\ \underline{R}(\underset{\sim}{i},o,t)$.

This follows by $1 \leftrightarrow 5$ of lemma 2. The following lemma will provide the basis for the decision method of SC. It was suggested by, and its proof is typical for, automata theory.

Lemma 5. There is an effective method for deciding truth of sentences $\underline{A}$ in Σ_1^{ω}.

Proof: Let $C(\underset{\sim}{r})$ be a formula of form $K[\underset{\sim}{r}(o)] \wedge (\forall t)\ H[\underset{\sim}{r}(t), \underset{\sim}{r}(t')] \wedge (\exists^{\omega} t)\ L[\underset{\sim}{r}(t)]$. Suppose $\underset{\sim}{r}$ is a k-tuple of predicates such that $\underline{C}(\underset{\sim}{r})$ holds. Then there are $x_1 < x_2 < \ldots$ such that $L[\underset{\sim}{r}(x_1)]$, $L[\underset{\sim}{r}(x_2)]$, ... Because $\underset{\sim}{r}$ has but a finite number of states, there must be a repetition $\underset{\sim}{r}(x_p) = \underset{\sim}{r}(x_q)$ of some state U.

Therefore, $(\exists \underset{\sim}{r})\ \underline{C}(\underset{\sim}{r})$ implies the assertion,

(1) There are words $\underline{x} = X_0X_1 \ldots X_a$ and $\underline{y} = Y_1Y_2 \ldots Y_b$ of states and a state U such that $L[U]$, and $K[X_0] \wedge H[X_0,X_1] \wedge \ldots \wedge H[X_{a-1},X_a] \wedge H[X_a,U]$, and $H[U,Y_1] \wedge H[Y_1,Y_2] \wedge \ldots \wedge H[Y_{b-1},Y_b] \wedge H[Y_b,U]$.

Conversely (1) implies $(\exists \underset{\sim}{r})\ \underline{C}(\underset{\sim}{r})$, because one has but to let $r = \underline{x}U\underline{y}U\underline{y}U\underline{y} \ldots$. Thus, a method I which decides, for given propositional formulas K,H,L and given state U, whether or not (1) holds, will also be a method for deciding truth of Σ_1^{ω}-sentences $(\exists r)\ \underline{C}(r)$. Clearly such a method I can be composed from a method II which, for given propositional formula $H[X,y]$ and given states V and W, decides whether or not,

(2) There is a word $\underline{x} = X_1X_2 \ldots X_a$ such that $H[V,X_1] \wedge H[X_1,X_2] \wedge \ldots \wedge H[X_{a-1},X_a] \wedge H[X_a,W]$.

Let $n = 2^k$ be the number of states, and note that in a word $\underline{x} = X_1X_2 \ldots X_a$ of length $a > n$ there must occur a repetition $X_p = X_q$, $p < q < a$. Clearly if $\underline{x}$ satisfies (2), then so does the shorter word $\underline{y} = X_1X_2 \ldots X_pX_{q+1}X_{q+2} \ldots X_a$ Therefore, to establish whether or not (2) holds it suffices to check among the finitely many words $\underline{x}$ of length $\leqq n$. This remark clearly yields a method II for (2), whereby lemma 5 is established.

4. Reduction of formulas of SC.

The following lemma is obtained by methods similar to those in Büchi [1], lemma 1.

Lemma 6. To every formula $\underline{A}(\underset{\sim}{i})$ of SC one can obtain an equivalent formula $\underline{B}(\underset{\sim}{i})$ belonging to some Σ_n (to some π_n).

Based on lemmas 1 to 4 one now can prove the fundamental fact on reduction of formulas in SC.

Lemma 7. To every formula $A(\underset{\sim}{i})$ in Σ_1^{ω} (in π_1^{ω}) one can obtain an equivalent formula $\underline{B}(\underset{\sim}{i})$ in π_1^{ω} (in Σ_1^{ω}).

Proof: Suppose $\underline{A}(i)$ is in Σ_1^{ω}, say

(1) $\underline{A}(\underset{\sim}{i})\ :\ (\exists r)\cdot K[\underset{\sim}{r}(o)] \wedge (\forall t)\ H[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \wedge (\exists^{\omega} t)\ L[\underset{\sim}{r}(t)]$.

If V,W are states of $\underset{\sim}{r}$ and $\underline{x} = X_0X_1 \ldots X_h$ is a word of states of $\underset{\sim}{i}$ then define,

$$[V,\underline{x},W]_1\ :\ \bigvee_{U_1 \ldots U_h} \cdot H[X_0,V,U_1] \wedge H[X_1,U_1,U_2] \wedge H[X_2,U_2,U_3] \wedge \ldots \wedge H[X_h,U_h,W]$$

$$[V,\underline{x},W]_2\ :\ \bigvee_{U_1 \ldots U_h} \cdot H[X_0,V,U] \wedge \ldots \wedge H[X_h,U_h,W] \wedge [L[U_1] \vee \ldots \vee L[U_h]].$$

(Read $[\ ]_1$ as "there is an H-transition through L vrom V by $\underline{x}$ to W.") Next define the binary relation $\backsim$ on words of states of $\underset{\sim}{i}$:

$$x \backsim \underline{y} \quad : \quad \bigwedge_{VW}([V,\underline{x},W]_1 \equiv [V,\underline{y},W]_1 \quad \bigwedge_{VW}([V,\underline{x},W]_2 \equiv [V,\underline{y},W]_2)$$

If m is the number of states of $\underset{\sim}{r}$, then clearly $\backsim$ is the intersection of $m^2 + m^2$ dichotomies. Therefore, (2) $\backsim$ is an equivalence relation of finite rank $a \leqq 2^{2m^2}$. Furthermore, using the definitions of $[\ ]_1$ and $[\ ]_2$ one obtains,

(3) $\backsim$ is a congruence relation on words. By (2), (3), and lemma 2 it follows that one can find formulas $\underline{E}_1(\underset{\sim}{i},x,y)$, ..., $\underline{E}_a(\underset{\sim}{i},x,y)$ such that

(4) $\underline{E}_1$, ..., $\underline{E}_a$ are regular formulas (i.e., belong to Σ_{reg}).

(5) $\underline{E}_1$, ..., $\underline{E}_a$ determine the congruence classes of $\backsim$.

Next one applies lemma 1 to the partition $\underline{E}_1$, ..., $\underline{E}_a$. If follows that for any $\underset{\sim}{i}$,

(6) $(\exists s)_\omega(\forall y)(\forall x)_o^y[s(x)s(6)\supset\underline{E}_1(\underset{\sim}{i},x,y)] \vee \ldots \vee (\exists s)_\omega(\forall y)(\forall x)_o^y[s(x)s(y)\supset\underline{E}_a(\underset{\sim}{i},x,y)]$.

If one defines for $1 \leqq c,d \leqq a$,

$$\underline{F}_{c,d}(i) \quad : \quad (\exists s)_\omega \cdot (\exists x)[s(x)\wedge \underline{E}_c(\underset{\sim}{i},o,x)]\wedge(\forall y)(\forall x)_o^y\ [s(x)s(y)\supset\underline{E}_d(\underset{\sim}{i},x,y)]$$

then clearly each disjunct of (6) is equivalent to a disjunction of $\underline{F}_{c,d}$'s. Therefore,

(7) $\bigvee_{1 \leqq c,d \leqq a} \underline{F}_{c,d}(\underset{\sim}{i})$, holds for all $\underset{\sim}{i}$.

Suppose now that $\underline{F}_{c,d}(\underset{\sim}{i})$ and $\underline{F}_{c,d}(\underset{\sim}{j})$. Then, by definition of $F_{c,d}$ and by (5) there are $x_1 < x_2 < x_3 < \ldots$ and $y_1 < y_2 < y_3 < \ldots$ such that

$$\underset{\sim}{i}(o) \ldots \underset{\sim}{i}(x_1-1) \backsim \underset{\sim}{j}(o) \ldots \underset{\sim}{j}(y_1-1)$$

$$\underset{\sim}{i}(x_p) \ldots \underset{\sim}{i}(x_{p+1}-1) \backsim \underset{\sim}{j}(y_p \ldots \underset{\sim}{j}(y_{p+1}-1) \qquad , p = 1,2,3,\ldots$$

Because of the definition of $\backsim$ and (1) it therefore follows that $A(\underset{\sim}{i}) = A(\underset{\sim}{j})$. Thus, if $\underline{F}_{c,d}(\underset{\sim}{i}) \wedge \underline{F}_{c,d}(\underset{\sim}{j})$ then $\underline{A}(\underset{\sim}{i}) \equiv \underline{A}(\underset{\sim}{j})$. Or restating this result,

(8) $(\forall i)[\underline{F}_{c,d}(\underset{\sim}{i}) \supset \underline{A}(\underset{\sim}{i})] \vee (\forall \underset{\sim}{i})[\underline{F}_{c,d}(\underset{\sim}{i}) \supset \sim A(\underset{\sim}{i})]$, for any $1 \leqq c,d \leqq a$.

If one now defines the set Φ of pairs $1 \leqq c,d \leqq a$ by,

(9) $\Phi(c,d) \equiv \sim (\exists \underset{\sim}{j})[A(\underset{\sim}{j}) \wedge \underline{F}_{c,d}(\underset{\sim}{j})]$, for $1 \leqq c,d \leqq a$

it follows by (7) and (8) that

(10) $\sim\underset{\sim}{\underline{A}}(\underset{\sim}{i}) \equiv \bigvee_{\Phi(c,d)} \underline{E}_{c,d}(i)$.

By (4), lemma 3, and lemma 4 it follows that there are formulas $\underline{D}_c(\underset{\sim}{i},s)$ and $\underline{G}_d(\underset{\sim}{i},s)$ in Σ_1 which are equivalent respectively to $(\exists x)[s(x) \wedge \underline{E}_c(\underset{\sim}{i},o,x)]$ and $(\forall y)(\forall x)_o^y[s(x)s(y) \supset \underline{E}_d(\underset{\sim}{i},x,y)]$. Referring to the definition of $\underline{F}_{c,d}$ this yields,

$$\underline{F}_{c,d}(\underset{\sim}{i}) \equiv (\exists s)\cdot(\exists^{\omega} t)\ s(t) \wedge \underline{D}_c(\underset{\sim}{i},s) \wedge \underline{G}_d(\underset{\sim}{i},s)$$

Because $\underline{D}_c$ and $\underline{G}_d$ are in Σ_1 it follows,

(11) $\underline{F}_{c,d}(\underset{\sim}{i}) \equiv (\exists s\underset{\sim}{p}\underset{\sim}{q}).\ I(o) \wedge (\forall t)\ J(t) \wedge (\exists t)\ M(t) \wedge (\exists t)\ N(t) \wedge (\exists^{\omega} t)\ s(t)$,

for some matrices $I[\underset{\sim}{p}(o),\underset{\sim}{q}(o)]$, $J[\underset{\sim}{i}(t), s(t), \underset{\sim}{q}(t), \underset{\sim}{p}(t), \underset{\sim}{q}(t'), \underset{\sim}{p}(t')]$, $M[\underset{\sim}{q}(t)]$, $N[\underset{\sim}{p}(t)]$. Note that $(\exists t)M(t) \wedge (\exists t)N(t) \wedge (\exists^{\omega} t)s(t)$ may be replaced by $[(\exists^{\omega} x)[(\exists t)_o^x M(t) \wedge (\exists t)_o^x N(t) \wedge s(x)]$, and this in turn may be replaced by $(\exists jh)[\sim j(o) \wedge \sim h(o) \wedge (\forall t)[j(t') \equiv j(t) \vee M(t)] \wedge (\forall t)[h(t') \equiv h(t) \vee N(t)] \wedge (\exists^{\omega} t)[j(t) \wedge h(t) \wedge s(t)]]$. The corresponding substitution in (11) then shows that, for any $1 \leq c,d \leq a$, the formula $\underline{F}_{c,d}(\underset{\sim}{i})$ is equivalent to a formula $\underline{J}_{c,d}(\underset{\sim}{i})$ in Σ_1^{ω}. By (10) it follows that $\sim\underline{A}(\underset{\sim}{i})$ is equivalent to a formula in Σ_1^{ω}, and therefore $\underline{A}(\underset{\sim}{i})$ is equivalent to a formula $\underline{B}(\underset{\sim}{i})$ in π_1^{ω}. This ends the proof of lemma 7.

<u>Theorem 1</u>. The hierarchy of relations on predicates definable by formulas of Σ_n, π_n collapses at n=2. To every formula $\underline{A}(\underset{\sim}{i})$ of SC one can find a formula $\underline{B}(\underset{\sim}{i})$ of Σ_1^{ω} (of π_1^{ω}, Σ_2, π_2) which is equivalent to $\underline{A}(\underset{\sim}{i})$.

Proof: Let $\underline{C}(\underset{\sim}{i})$: $(\exists r)[K(o) \wedge (\forall t)\ H(t) \wedge (\exists t)\ L(t)]$ be any formula of Σ_1. Then $\underline{C}(\underset{\sim}{i}) \equiv (\exists \underset{\sim}{r})[K(o) \wedge (\forall t)\ H(t) \wedge (\exists^{\omega} x)(\exists t)_o^x\ L(t)]$. If the term $(\exists t)_o^x\ L(t)$ is dealt with as shown at the end of the proof of lemma 7, this yields a formula $\underline{D}(\underset{\sim}{i})$ in Σ_1^{ω} which is equivalent to $C(\underset{\sim}{i})$. Using this remark part one of theorem 1 easily follows by an induction of n and the use of lemma 7. By lemma 6 the other part of theorem 1 follows.

Remarks: 1. The set $\underline{\underline{U}}$ consisting of all infinite i can be defined by a Σ_1^{ω} (a Σ_2) formula, but not by a π_1^{ω} (a π_2) formula. This shows that theorem 1 cannot be much improved.

2. Using theorem 1 one easily shows that also formulas $\underline{A}(\underset{\sim}{i},x_1, \ldots, x_n)$ of SC, containing individual variables can be put into a normal form, namely

$$(\exists \underset{\sim}{r})\cdot K[\underset{\sim}{r}(o)] \wedge (\forall t)\ H[\underset{\sim}{i}(t), \underset{\sim}{r}(t), \underset{\sim}{r}(t')] \wedge (\exists^{\omega} t)\ L[\underset{\sim}{r}(t)] \wedge U[\underset{\sim}{r}(x_1)] \wedge \cdots \wedge U[\underset{\sim}{r}(x_n)]$$

This yields rather complete information on definability in SC. For example,

3. A conjecture of Robinson [10]: A relation $\underline{\underline{R}}(x_1, \ldots, x_n)$ on natural numbers is definable in SC if and only if it is definable in SC_{fin}, which is like SC except that the variables i, j, r, ... range over finite sets of natural numbers. This follows from remark 2 by methods similar to those in the proof of lemma 5. For complete discussion of definability in SC_{fin} see Büchi [1].

4. A relation $\underline{\underline{R}}(i_1, \ldots, i_n)$ on finite sets of natural numbers is definable in SC if and only if it is definable in SC_{fin}.

5. Analyzing the proof of lemma 7 one obtains: A set $\underline{\underline{R}}$ of n-tuples $\underset{\sim}{i}$ of predacates is definable in SC if and only if $\underline{\underline{R}} = \underline{S}_1 \cup \ldots \cup \underline{S}_k$ whereby each $\underline{S}_c$ is of the form $\underline{\underline{A}}\frown\underline{\underline{B}}\frown\underline{\underline{B}}\frown\underline{\underline{B}}\frown\ldots$, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ being regular sets of words.

Lemmas 2, 3, 4, 6 can be stated and proved in a strong constructive version. To see that this also holds for lemma 7, it remains to ascertain that the finite set Φ of pairs, defined by (9) in the proof of 7, may be obtained effectively for a given $\underline{A}(\underset{\sim}{i})$. This follows by lemma 5, if one observes that $\underline{A}(\underset{\sim}{i})$, $\underline{F}_{c,d}(\underset{\sim}{i})$ and therefore $(\exists j)[\underline{A}(j) \wedge \underline{F}_{c,d}(j)]$ are equivalent to Σ_1^ω formulas. It now follows that theorem 1 can also be proved in an effective version. In particular, to every sentence $\underline{A}$ of SC one can effectively construct an equivalent one belonging to Σ_1^ω. Applying lemma 5 again this yields,

<u>Theorem 2</u>. There is an effective method for deciding truth of sentences in SC.

The strength of these results is best seen by noting some very special cases which occur in the literature and have been obtained by rather divergent methods:

1. The decidability of Σ_2-sentences of SC contains the result of Friedman [3], and implies the existence of various other algorithms of finite automata theory as programmed by Church [2]. It also implies some of the results of Wang [11].

2. In SC one can define $x = y$, $x < y$, $x \equiv y \pmod{k}$ (for k=1,2,...). The decidability of SC therefore considerably improves a result of Putnam [7].

3. In SC one can define "i is finite." Theorem 2 therefore implies the decidability of SC_{fin}, which was also proved in Büchi [1], and according to Robinson [10] is due to A. Ehrenfeucht.

4. The decidability of the first order theory of [Nn, +, Pw] follows from theorem 4 and improves the classical result of Presburger.

5. Theorem 2 is closely related to another classical result, namely the decidability of the monadic predicate calculus of second order, proved first by Th. Skolem and later by H. Behmann. A modified form of lemma 6 yields a rather simple solution to this problem.

BIBLIOGRAPHY

1. J. R. Büchi, "Weak second order arithmetic and finite automata." Zeitschrift für Math. Log. und Grundl. der Math. (1960), pp.

2. Alonzo Church, "Application of Recursive Arithmetic to the Problem of Circuit Synthesis," Notes of the Summer Institute of Symbolic Logic, Cornell, 1957, pp. 3-50, and "Application of Recursive Arithmetic in the Theory of Computing and Automata," Notes: Advanced Theory of the Logical Design of Digital Computers, U. of Michigan Summer Session, 1959.

3. Joyce Friedman, "Some Results in Church's Restricted Recursive Arithmetic," Journal of Symbolic Logic, 22, pp. 337-342 (1957).

4. S. C. Kleene, "Representation of Events in Nerve Nets and Finite Automata." Automata Studies, Princeton Univ.Press, 1956, pp. 3-41.

5. Denes König, Theorie der endlichen und unendlichen Graphen. Akad. Verlagsges., Leipzig, 1936.

6. John Myhill, "Finite Automata and Representation of Events," WADC Report TR 57-624 Fundamental Concepts in the Theory of Systems, October 1957, pp. 112-137.

7. Hillary Putnam, "Decidability and essential undecidability." Journal of Symbolic Logic, 22 (1957), pp. 39-54.

8. M. Rabin and D. Scott, "Finite Automata and Their Decision Problems." IBM Journal, April, 1959, pp. 114-125.

9. F. P. Ramsey, "On a problem of formal logic." Proc. London Math. Soc. (2) 30 (1929), pp. 264-286.

10. R. M. Robinson, "Restricted set-theoretical definitions in arithmetic." Proc. Am. Math. Soc., 9 (1958), pp. 238-242.

11. Hao Wang, "Circuit synthesis by solving sequential Boolean equations." Zeitsch. für Math. Logik und Grundl. der Math., 5 (1959), pp. 291-322.